Pergamon

Int. J. Solids Structures Vol. 32, No. 1, pp. 127–133, 1995

0020–7683/95 $9.50 + .00

0020–7683(94)00134–0

LETTER TO THE EDITOR

Discussion of "An elastodynamic solution for an anisotropic hollow sphere", *Int. J. Solids Structures*, Vol. 31, No. 7, pp. 903–911 (1994).

In this paper, Wang has presented an analytical solution to the problem of impact response of a spherically symmetric, transversely isotropic hollow sphere. To this end, he has employed a finite Hankel transform whose kernel function is the eigenfunction $G(\xi_i, r)$ determined from free vibration analysis of the associated problem [eqn (18c) in the original paper]. Numerical results are presented for the cases of a uniform sudden load and an exponential decaying shock load.

For the case of a suddenly applied uniform load, Pao and Ceranoglu (1978) presented an exact solution to an isotropic hollow sphere by using the ray theory, and Bickford and Warren (1967) obtained the solution to a transversely isotropic hollow sphere by using Laplace transforms with rational approximations for inverse Laplace transforms. On the other hand, we have recently analysed the impact response of axisymmetric multi-layered isotropic hollow spheres (Kobayashi *et al.*, 1994) and transversely isotropic hollow cylinders and spheres (Ishimaru *et al.*, 1994) by means of the method of eigenfunction expansion developed by Reismann (1967). Although the material constants used are somewhat different from each other, we can compare the numerical results presented with each other, especially for the isotropic case. As a result, we can understand that Wang's response curves shown in Figs 1–3 are incorrect.

In the following, we show our solution procedure for the transversely isotropic hollow spheres and the correct numerical results for both the isotropic and transversely isotropic cases.

For a transversely isotropic sphere, the nonzero stress components $\sigma_r(r, t)$ and $\sigma_\theta(r, t)$ are expressed in terms of the radial displacement $U(r, t)$ as

$$\sigma_r = A_{11}\frac{\partial U}{\partial r} + 2A_{12}\frac{U}{r}, \qquad \sigma_\theta = A_{12}\frac{\partial U}{\partial r} + (A_{22} + A_{23})\frac{U}{r}, \tag{1}$$

where A_{ij} are material constants given by eqn (3) in the original paper. The displacement equation of motion is given by

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} - A^2\frac{U}{r^2} = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2}, \tag{2}$$

where $A^2 = 2(A_{22} + A_{23} - A_{12})/A_{11}$ and $V = \sqrt{(A_{11}/\rho)}$.

Following Reismann (1967), a solution of eqn (2) can be taken in the form

$$U(r, t) = U_s(r, t) + \sum_{i=1}^{\infty} Q_i(t)U_i(r), \tag{3}$$

where $U_s(r, t)$ and $U_i(r)$ are, respectively, the quasi-static solution and eigenfunction, and $Q_i(t)$ is a function of time t only to be determined from the initial conditions.

The quasi-static solution $U_s(r, t)$ satisfies the equation

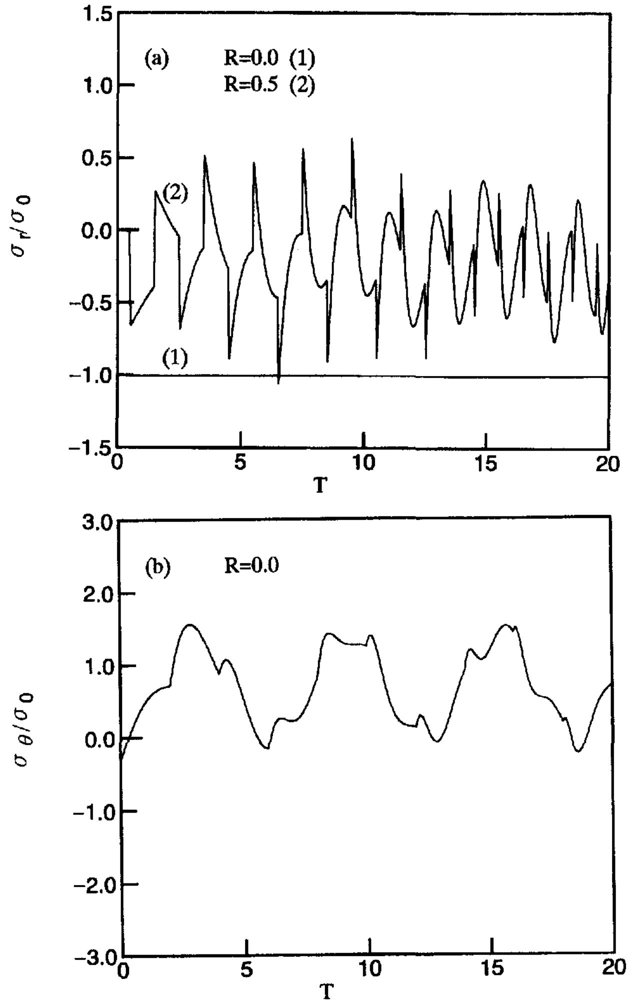

Fig. 1. The isotropic hollow sphere for the case of $E_r = E_\theta = 200$ GPa and $\alpha = 0$: (a) σ_r at $R = 0.0$ and 0.5; (b) σ_θ at $R = 0.0$.

$$\frac{\partial^2 U_s}{\partial r^2}+\frac{2}{r}\frac{\partial U_s}{\partial r}-A^2\frac{U_s}{r^2}=0 \tag{4}$$

subject to the following associated boundary conditions:

$$A_{11}\frac{\partial U_s}{\partial r}+2A_{12}\frac{U_s}{r}=\psi_1(t)\quad \text{at } r=a;\qquad \psi_2(t)\quad \text{at } r=b. \tag{5}$$

The explicit form of $U_s(r, t)$ is presented in eqns (8) and (9) in the original paper.

The eigenfunction $U_i(r)$ is obtained from the free vibration analysis and satisfies the equation

$$\frac{\partial^2 U_i}{\partial r^2}+\frac{2}{r}\frac{\partial U_i}{\partial r}+\left[\xi_i^2-\frac{A^2}{r^2}\right]U_i=0 \tag{6}$$

subject to the following stress-free boundary conditions:

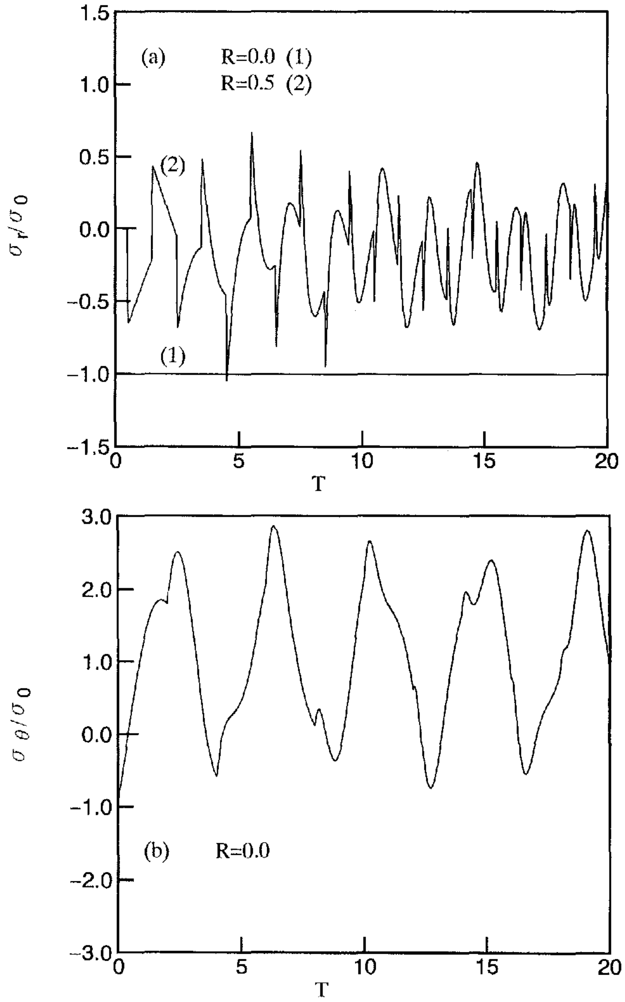

Fig. 2. The anisotropic hollow sphere for the case of $E_r = 200$ GPa, $E_\theta/E_r = 25/9$ and $\alpha = 0$: (a) σ_r at $R = 0.0$ and 0.5; (b) σ_θ at $R = 0.0$; (c) σ_θ at $R = 0.5$; (d) σ_θ at $R = 1.0$.

$$A_{11}\frac{\partial U_i}{\partial r}+2A_{12}\frac{U_i}{r}=0 \quad \text{at } r=a \text{ and } r=b, \tag{7}$$

where $\xi_i = \omega_i/V$, ω_i being a natural circular frequency.

The eigenfunction satisfying eqn (6) and corresponding frequency equation determining the ith eigenvalue ξ_i are given as follows:

$$U_i(r)=\frac{1}{\sqrt{\xi_i r}}\left[Y_a J_k(\xi_i r)-J_a Y_k(\xi_i r)\right] \tag{8}$$

$$Y_a J_b - Y_b J_a = 0, \tag{9}$$

where

$$k^2=\tfrac{1}{4}+A^2 \tag{10}$$

[the expression of k in eqn (13a) in the original paper is incorrect] and the contents of J_a, Y_a, J_b and Y_b are presented by eqns (15b–g) in the original paper. It should be noted that

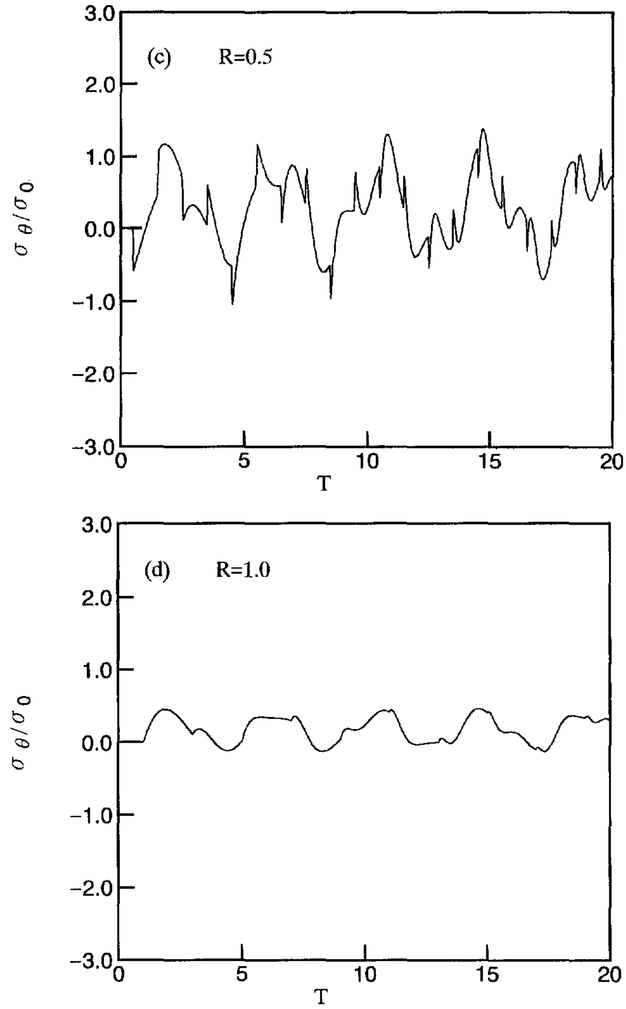

Fig. 2—*Continued.*

the eigenfunction $U_i(r)$ is related with the kernel function $G(\xi_i, r)$ of eqn (18c) used by Wang as $U_i(r) = (\xi_i r)^{1/2} G(\xi_i r)$.

With the aid of the reciprocal theorem, the orthogonality of the eigenfunction can be proved to be

$$\int_a^b U_i(r) U_j(r)\, \mathrm{d}v = \delta_{ij} N_i^2, \tag{11}$$

where $\mathrm{d}v = 4\pi r^2\, \mathrm{d}r$, δ_{ij} is the Kronecker's delta, and N_i is the norm of the eigenfunction.

Substituting eqn (3) into eqn (2) and utilizing eqns (4) and (6) and the orthogonality of eigenfunction, we obtain the equation for the determination of $Q_i(t)$ as

$$\ddot{Q}_i(t) + \omega_i^2 Q_i(t) = \ddot{\phi}_i(t), \tag{12}$$

where

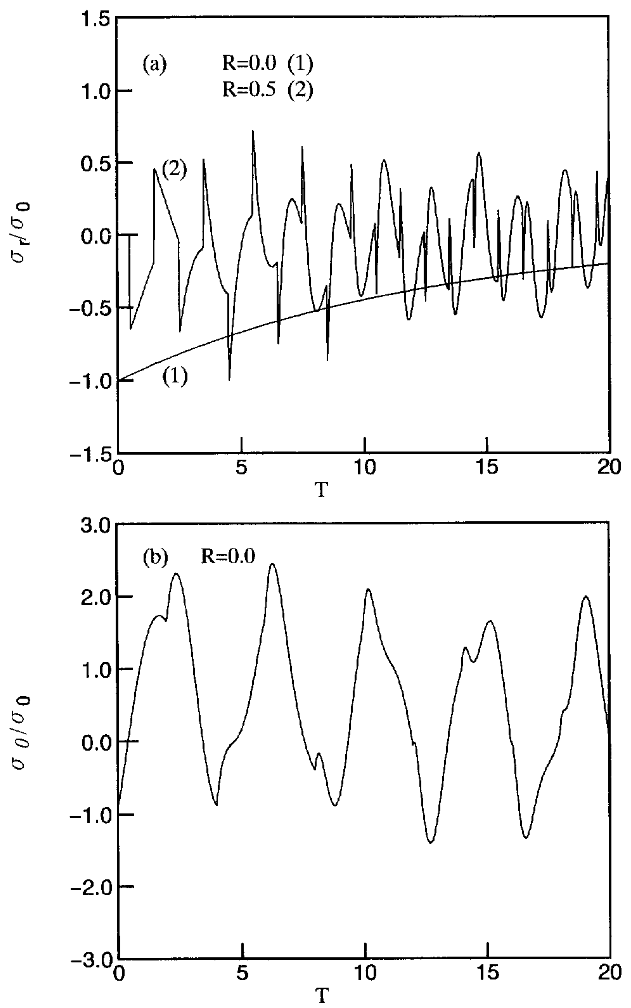

Fig. 3. The anisotropic hollow sphere for the case of $E_r = 200$ GPa, $E_\theta/E_r = 25/9$ and $\alpha = 500$: (a) σ_r at $R = 0.0$ and 0.5; (b) σ_θ at $R = 0.0$; (c) σ_θ at $R = 0.5$; (d) σ_θ at $R = 1.0$.

$$\phi_i(t) = -\frac{1}{N_i^2}\int_a^b U_s(r,t)U_i(r)\,\mathrm{d}v. \tag{13}$$

The solution of eqn (12) is given by

$$Q_i(t) = Q_i(0)\cos\omega_i t + \frac{1}{\omega_i}\dot{Q}(0)\sin\omega_i t + \frac{1}{\omega_i}\int_0^t \ddot{\phi}_i(\tau)\sin[\omega_i(t-\tau)]\,\mathrm{d}\tau, \tag{14}$$

where $Q_i(0)$, $\dot{Q}_i(0)$ are constants of integration to be determined from the initial conditions:

$$U(r,0) = \dot{U}(r,0) = 0,$$

i.e.

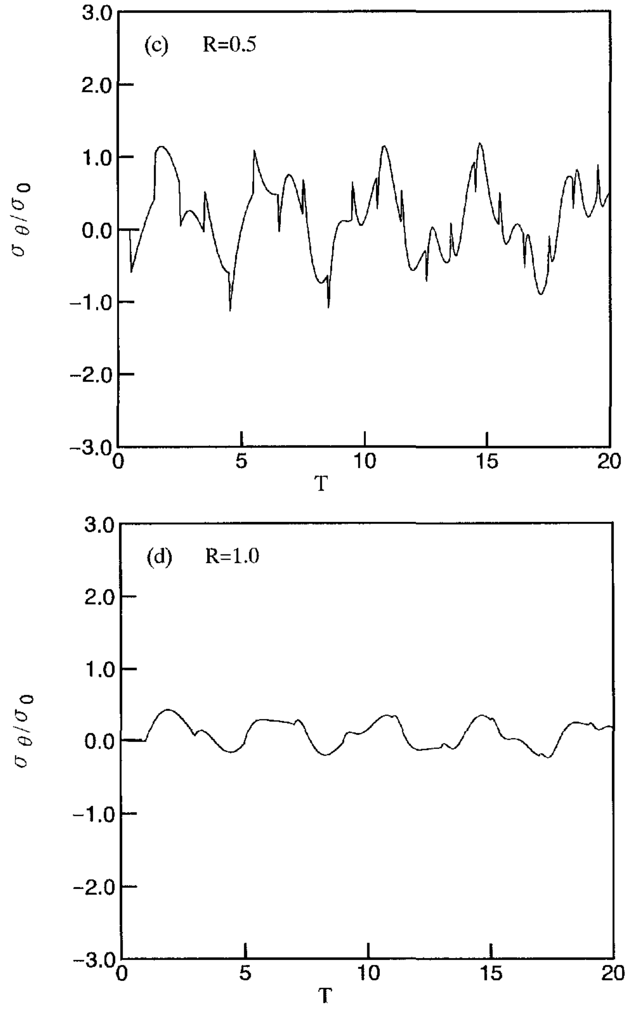

Fig. 3—*Continued.*

$$\sum_{i=1}^{\infty} Q_i(0)U_i(r) = -U_s(r,0), \quad \sum_{i=1}^{\infty} \dot{Q}_i(0)U_i(r) = -\dot{U}_s(r,0). \tag{15}$$

Making use of the orthogonality of eigenfunction, we obtain the following results from the above equations as

$$Q_i(0) = \phi_i(0), \quad \dot{Q}_i(0) = \dot{\phi}_i(0). \tag{16}$$

Substitution of these into eqn (14) and integration by parts gives

$$Q_i(t) = \phi_i(t) - \omega_i \int_0^t \phi_i(\tau) \sin[\omega_i(t-\tau)]\, d\tau. \tag{17}$$

Thus our solution form is identical to that in the original paper.

Numerical results with the same values of material constants used by Wang are presented in Figs 1–3.

Finally, we present the correct expressions for eqns (10d, e) and (12d, e) in the original paper as follows:

$$U_d(r,0) = -U_s(r,0), \qquad \dot{U}_d(r,0) = -\dot{U}_s(r,0)$$

for eqns (10d, e) and

$$f(r,0) = -U_{s1}(r,0), \quad \dot{f}(r,0) = -\dot{U}_{s1}(r,0)$$

for eqns (12d, e).

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